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On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces

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Abstract
In this paper, we introduce and study iterative schemes for solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces and prove that the modified Mann iteration converges weakly to a common solution of the considered problems. Moreover, we present some examples and numerical results for the main results.

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Keywords: nonspreading multi-valued mapping; split equilibrium problem; fixed point problem; weak convergence; Hilbert space

1 Introduction

In the following, let H_1 and H_2 be real Hilbert spaces with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty subset of H_1. The *equilibrium problem* is to find a point $\hat{x}\in C$ such that

$$F_1(\hat{x},y)\geq 0 \tag{1.1}$$

for all $y\in C$. Since its inception by Blum and Oettli [1] in 1994, the equilibrium problem (1.1) has received much attention due to its applications in a large variety of problems arising in numerous problems in physics, optimizations, and economics. Some methods have been rapidly established for solving this problem (see [2–5]).

Very recently, Kazmi and Rizvi [6] introduced and studied the following split equilibrium problem:

Let $C\subseteq H_1$ and $Q\subseteq H_2$. Let $F_1: C\times C\to\mathbb{R}$ and $F_2: Q\times Q\to\mathbb{R}$ be two bifunctions. Let $A: H_1\to H_2$ be a bounded linear operator. The *split equilibrium problem* is to find $\hat{x}\in C$ such that

$$F_1(\hat{x},x)\geq 0 \quad \text{for all } x\in C \tag{1.2}$$

and such that

$$\hat{y}=A\hat{x}\in Q \text{ solves } F_2(\hat{y},y)\geq 0 \quad \text{for all } y\in Q. \tag{1.3}$$

Note that the problem (1.2) is the classical equilibrium problem and we denote its solution set by $EP(F_1)$. The inequalities (1.2) and (1.3) constitute a pair of equilibrium problems which have to find the image $\hat{y} = A\hat{x}$, under a given bounded linear operator A, of the solution $\hat{x}$ of (1.2) in H_1 is the solution of (1.3) in H_2. We denote the solution set of (1.3) by $EP(F_2)$. The solution set of the split equilibrium problem (1.2) and (1.3) is denoted by $\Omega = \{z \in EP(F_1) : Az \in EP(F_2)\}$.

A subset $C \subset H_1$ is said to be *proximinal* if, for each $x \in H_1$,

$$\|x - y\| = d(x, C) = \inf\bigl\{\|x - z\| : z \in C\bigr\}.$$

Let $CB(C)$, $K(C)$, and $P(C)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal bounded subset of C, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A, B) = \max\Bigl\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\Bigr\}$$

for all $A, B \in CB(C)$ where $d(x, B) = \inf_{b \in B}\|x - b\|$. An element $p \in C$ is called a *fixed point* of $T : C \to CB(C)$ if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$. We say that $T : C \to CB(C)$ is:

(1) *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|$$

for all $x, y \in C$;

(2) *quasi-nonexpansive* if

$$H(Tx, Tp) \leq \|x - p\|$$

for all $x \in C$ and $p \in F(T)$.

Recently, the existence of fixed points and the convergence theorems of multi-valued mappings have been studied by many authors (see [7–12]).

Hussain and Khan [13] presented the fixed point theorems of a *-nonexpansive multi-valued mapping and the strong convergence of its iterates to a fixed point defined on a closed and convex subset of a Hilbert space by using the best approximation operator $P_T x$, which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$. The convergence theorems and its applications in this direction have been established by many authors (for instance, see [10, 14, 15]).

In 2011, Song and Cho [16] gave the example of a multi-valued mapping T which is not necessary nonexpansive, but P_T is nonexpansive. This is an important tool for studying the fixed point theory for multi-valued mappings.

Kohsaka and Takahashi [17] introduced a class of mappings which is called *nonspreading mapping*. Let C be a subset of Hilbert spaces H_1. A mapping $T : C \to C$ is said to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. Subsequently, Iemoto and Takahashi [18] showed that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$.

Very recently, Liu [19] introduced the following class of multi-valued mappings: a multi-valued mapping $T : C \to CB(C)$ is said to be *nonspreading* if

$$2\|u_x - u_y\|^2 \le \|u_x - y\|^2 + \|u_y - x\|^2$$

for some $u_x \in Tx$ and $u_y \in Ty$ for all $x, y \in C$. He proved a weak convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points.

In this paper, we introduce, by using Hausdorff metric, the class of nonspreading multi-valued mappings. We say that a mapping $T : C \to CB(C)$ is a *k-nonspreading multi-valued mapping* if there exists $k > 0$ such that

$$H(Tx, Ty)^2 \le k\big(d(Tx, y)^2 + d(x, Ty)^2\big) \tag{1.4}$$

for all $x, y \in C$.

It is easy to see that, if T is $\frac{1}{2}$-nonspreading, then T is nonspreading in the case of single-valued mappings (see [17, 20]). Moreover, if T is a $\frac{1}{2}$-nonspreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Indeed, for all $x \in C$ and $p \in F(T)$, we have

$$\begin{aligned} 2H(Tx, Tp)^2 &\le d(Tx, p)^2 + d(x, Tp)^2 \\ &\le H(Tx, Tp)^2 + \|x - p\|^2. \end{aligned}$$

It follows that

$$H(Tx, Tp) \le \|x - p\|. \tag{1.5}$$

We now give an example of a $\frac{1}{2}$-nonspreading multi-valued mapping which is not nonexpansive.

Example 1.1 Consider $C = [-3, 0]$ with the usual norm. Define $T : C \to CB(C)$ by

$$Tx = \begin{cases} \{0\}, & x \in [-2, 0]; \\ [-\frac{|x|}{|x|+1}, 0], & x \in [-3, -2). \end{cases}$$

Now, we show that T is $\frac{1}{2}$-nonspreading. In fact, we have the following cases:

Case 1: If $x, y \in [-2, 0]$, then $H(Tx, Tx) = 0$.

Case 2: If $x \in [-2, 0]$ and $y \in [-3, -2)$, then $Tx = \{0\}$ and $Ty = [-\frac{|y|}{|y|+1}, 0]$. This implies that

$$2H(Tx, Ty)^2 = 2\left(\frac{|y|}{|y| + 1}\right)^2 < 2 < y^2 \le d(Tx, y)^2 + d(x, Ty)^2.$$

Case 3: If $x, y \in [-3,-2)$, then $Tx = [-\frac{|x|}{|x|+1}, 0]$ and $Ty = [-\frac{|y|}{|y|+1}, 0]$. This implies that

$$2H(Tx, Ty)^2 = 2\left(\frac{|x|}{|x|+1} - \frac{|y|}{|y|+1}\right)^2 < 2 < d(Tx, y)^2 + d(x, Ty)^2.$$

On the other hand, T is not nonexpansive since for $x = -2$ and $y = -\frac{5}{2}$, we have $Tx = \{0\}$ and $Ty = [-\frac{5}{7}, 0]$. This shows that $H(Tx, Ty) = \frac{5}{7} > \frac{1}{2} = |-2-(-\frac{5}{2})| = \|x-y\|$.

In 1953, Mann [21] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n \tag{1.6}$$

for each $n \in \mathbb{N}$, where the initial point x_1 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0,1]$.

Motivated by the previous results, in this paper, we introduce and study the Mann-type iteration to approximate a common solution of the split equilibrium problem and the fixed point problem for a $\frac{1}{2}$-nonspreading multi-valued mapping and prove some weak convergence theorems in Hilbert spaces. Finally, we give some examples and numerical results to illustrate our main results.

2 Preliminaries

We now provide some results for the main results. In a Hilbert space H_1, let C be a nonempty closed convex subset of H_1. For every point $x \in H_1$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H_1 onto C. We know that P_C is a firmly nonexpansive mapping from H_1 onto C, *i.e.*,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H_1.$$

Further, for any $x \in H_1$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C.$$

A mapping $A : C \to H_1$ is called *α-inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Lemma 2.1 *Let H_1 be a real Hilbert space. Then the following equations hold*:

(1) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in H_1$;

(2) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ *for all* $x, y \in H_1$;

(3) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $t \in [0,1]$ *and* $x, y \in H_1$;

(4) *If* $\{x_n\}_{n=1}^\infty$ *is a sequence in* H_1 *which converges weakly to* $z \in H_1$, *then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2$$

for all $y \in H_1$.

A space X is said to satisfy *Opial's condition* if, for any sequence x_n with $x_n \rightharpoonup x$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known that every Hilbert space satisfies Opial's condition.

Lemma 2.2 [22] *Let X be a Banach space which satisfies Opial's condition and $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Assumption 2.3 [1] Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(1) $F_1(x,x) = 0$ for all $x \in C$;
(2) F_1 is monotone, *i.e.*, $F_1(x,y) + F_1(y,x) \leq 0$ for all $x \in C$;
(3) for each $x,y,z \in C$, $\limsup_{t\to 0^+} F_1(tz + (1-t)x, y) \leq F_1(x,y)$;
(4) for each $x \in C$, $y \to F_1(x,y)$ is convex and lower semi-continuous.

Lemma 2.4 [3] *Assume that $F_1 : C \times C \to \mathbb{R}$ satisfies Assumption* 2.3. *For any $r > 0$ and $x \in H_1$, define a mapping $T_r^{F_1} : H_1 \to C$ as follows*:

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

Then the following hold:

(1) $T_r^{F_1}$ *is nonempty and single-valued*;
(2) $T_r^{F_1}$ *is firmly nonexpansive, i.e., for any $x, y \in H_1$,*

$$\left\| T_r^{F_1}x - T_r^{F_1}y \right\|^2 \leq \left\langle T_r^{F_1}x - T_r^{F_1}y, x - y \right\rangle;$$

(3) $F(T_r^{F_1}) = EP(F_1)$;
(4) $EP(F_1)$ *is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.3. For each $s > 0$ and $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows:

$$T_s^{F_2}(w) = \left\{ d \in Q : F_2(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \forall e \in Q \right\}.$$

Then we have the following:

(5) $T_s^{F_2}$ is nonempty and single-valued;
(6) $T_s^{F_2}$ is firmly nonexpansive;
(7) $F(T_s^{F_2}) = EP(F_2, Q)$;
(8) $EP(F_2, Q)$ is closed and convex.

Condition (A) Let H_1 be a Hilbert space and C be a subset of H_1. A multi-valued mapping $T : C \to CB(C)$ is said to satisfy *Condition* (A) if $\|x - p\| = d(x, Tp)$ for all $x \in H_1$ and $p \in F(T)$.

Remark 2.5 We see that T satisfies Condition (A) if and only if $Tp = \{p\}$ for all $p \in F(T)$. It is well known that the best approximation operator P_T, which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$, also satisfies Condition (A).

3 Main results

Now, we are ready to prove some weak convergence theorem for $\frac{1}{2}$-nonspreading multi-valued mappings in Hilbert spaces. To this end, we need the following crucial results.

Lemma 3.1 *Let C be a closed and convex subset of a real Hilbert space H_1 and $T : C \to K(C)$ be a k-nonspreading multi-valued mapping such that $k \in (0, \frac{1}{2}]$. If $x, y \in C$ and $a \in Tx$, then there exists $b \in Ty$ such that*

$$\|a - b\|^2 \leq H(Tx, Ty)^2 \leq \frac{k}{1-k}\bigl(\|x - y\|^2 + 2\langle x - a, y - b\rangle\bigr).$$

Proof Let $x, y \in C$ and $a \in Tx$. By Nadler's theorem (see [8]), there exists $b \in Ty$ such that

$$\|a - b\|^2 \leq H(Tx, Ty)^2.$$

It follows that

$$\begin{aligned}
&\frac{1}{k} H(Tx, Ty)^2 \\
&\quad \leq d(Tx, y)^2 + d(x, Ty)^2 \\
&\quad \leq \|a - y\|^2 + \|x - b\|^2 \\
&\quad \leq \|a - x\|^2 + 2\langle a - x, x - y\rangle + \|x - y\|^2 + \|x - a\|^2 + 2\langle x - a, a - b\rangle + \|a - b\|^2 \\
&\quad = 2\|a - x\|^2 + \|x - y\|^2 + \|a - b\|^2 + 2\bigl\langle a - x, x - a - (y - b)\bigr\rangle \\
&\quad \leq 2\|a - x\|^2 + \|x - y\|^2 + H(Tx, Ty)^2 + 2\bigl\langle a - x, x - a - (y - b)\bigr\rangle.
\end{aligned}$$

This implies that

$$H(Tx, Ty)^2 \leq \frac{k}{1-k}\bigl(\|x - y\|^2 + 2\langle x - a, y - b\rangle\bigr).$$

This completes the proof. □

Lemma 3.2 *Let C be a closed and convex subset of a real Hilbert space H_1 and $T : C \to K(C)$ be a k-nonspreading multi-valued mapping such that $k \in (0, \frac{1}{2}]$. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

Proof Let $\{x_n\}$ be a sequence in C which converges weakly to p and let $y_n \in Tx_n$ be such that $\|x_n - y_n\| \to 0$.

Now, we show that $p \in F(T)$. By Lemma 3.1, there exists $z_n \in Tp$ such that

$$\|y_n - z_n\|^2 \leq \frac{k}{1-k}\bigl(\|x_n - p\|^2 + 2\langle x_n - y_n, p - z_n\rangle\bigr).$$

Since Tp is compact and $z_n \in Tp$, there exists $\{z_{n_i}\} \subset \{z_n\}$ such that $z_{n_i} \to z \in Tp$. Since $\{x_n\}$ converges weakly, it is bounded. For each $x \in H_1$, define a function $f : H_1 \to [0,\infty)$ by

$$f(x) := \limsup_{i\to\infty} \frac{k}{1-k} \|x_{n_i} - x\|^2.$$

Then, by Lemma 2.1(4), we obtain

$$f(x) = \limsup_{i\to\infty} \frac{k}{1-k} \bigl(\|x_{n_i} - p\|^2 + \|p - x\|^2\bigr)$$

for all $x \in H_1$. Thus $f(x) = f(p) + \frac{k}{1-k}\|p - x\|^2$ for all $x \in H_1$. It follows that

$$f(z) = f(p) + \frac{k}{1-k} \|p - z\|^2. \tag{3.1}$$

We observe that

$$\begin{aligned} f(z) &= \limsup_{i\to\infty} \frac{k}{1-k} \|x_{n_i} - z\|^2 = \limsup_{i\to\infty} \frac{k}{1-k} \|x_{n_i} - y_{n_i} + y_{n_i} - z\|^2 \\ &\le \limsup_{i\to\infty} \frac{k}{1-k} \|y_{n_i} - z\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} f(z) &\le \limsup_{i\to\infty} \frac{k}{1-k} \|y_{n_i} - z\|^2 \\ &= \limsup_{i\to\infty} \frac{k}{1-k} \bigl(\|y_{n_i} - z_{n_i} + z_{n_i} - z\|\bigr)^2 \\ &\le \limsup_{i\to\infty} \frac{k}{1-k} \bigl(\|x_{n_i} - p\|^2 + 2\langle x_{n_i} - y_{n_i}, p - z_{n_i}\rangle\bigr) \\ &\le \limsup_{i\to\infty} \frac{k}{1-k} \|x_{n_i} - p\|^2 \\ &= f(p). \end{aligned} \tag{3.2}$$

Hence it follows from (3.1) and (3.2) that $\|p - z\| = 0$. This completes the proof. □

Theorem 3.3 *Let H_1, H_2 be two real Hilbert space and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to K(C)$ a $\frac{1}{2}$-nonspreading multi-valued mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption* 2.3 *and F_2 is upper semi-continuous in the first argument. Assume that T satisfies Condition* (A) *and $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1)$ and $Az \in EP(F_2)\}$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} \in \alpha_n x_n + (1-\alpha_n)Tu_n, \end{cases} \tag{3.3}$$

for all $n \ge 1$, where $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0,\infty)$, and $\gamma \in (0,1/L)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(1) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(2) $\liminf_{n\to\infty} r_n > 0$.
Then the sequence $\{x_n\}$ *defined by* (3.3) *converges weakly to* $p \in \Theta$.

Proof We first show that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping. Since $T_{r_n}^{F_2}$ is firmly nonexpansive and $I - T_{r_n}^{F_2}$ is 1-inverse strongly monotone, we see that

$$\begin{aligned}\left\|A^*\left(I - T_{r_n}^{F_2}\right)Ax - A^*\left(I - T_{r_n}^{F_2}\right)Ay\right\|^2 &= \left\langle A^*\left(I - T_{r_n}^{F_2}\right)(Ax - Ay), A^*\left(I - T_{r_n}^{F_2}\right)(Ax - Ay)\right\rangle \\ &= \left\langle\left(I - T_{r_n}^{F_2}\right)(Ax - Ay), AA^*\left(I - T_{r_n}^{F_2}\right)(Ax - Ay)\right\rangle \\ &\le L\left\langle\left(I - T_{r_n}^{F_2}\right)(Ax - Ay), \left(I - T_{r_n}^{F_2}\right)(Ax - Ay)\right\rangle \\ &= L\left\|\left(I - T_{r_n}^{F_2}\right)(Ax - Ay)\right\|^2 \\ &\le L\left\langle Ax - Ay, \left(I - T_{r_n}^{F_2}\right)(Ax - Ay)\right\rangle \\ &= L\left\langle x - y, A^*\left(I - T_{r_n}^{F_2}\right)Ax - A^*\left(I - T_{r_n}^{F_2}\right)Ay\right\rangle\end{aligned}$$

for all $x, y \in H_1$. This implies that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping. Since $\gamma \in (0, \frac{1}{L})$, it follows that $I - \gamma A^*(I - T_{r_n}^{F_2})A$ is nonexpansive.

Now, we divide the proof into six steps as follows:

Step 1. Show that $\{x_n\}$ is bounded.

Let $p \in \Theta$. Then $p = T_{r_n}^{F_1}p$ and $(I - \gamma A^*(I - T_{r_n}^{F_2})A)p = p$. Thus we have

$$\begin{aligned}\|u_n - p\| &= \left\|T_{r_n}^{F_1}\left(I - \gamma A^*\left(I - T_{r_n}^{F_2}\right)A\right)x_n - T_{r_n}^{F_1}\left(I - \gamma A^*\left(I - T_{r_n}^{F_2}\right)A\right)p\right\| \\ &\le \left\|\left(I - \gamma A^*\left(I - T_{r_n}^{F_2}\right)A\right)x_n - \left(I - \gamma A^*\left(I - T_{r_n}^{F_2}\right)A\right)p\right\| \\ &\le \|x_n - p\|.\end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n\|x_n - p\| + (1 - \alpha_n)\|z_n - p\| \quad \text{for some } z_n \in Tu_n \\ &\le \alpha_n\|x_n - p\| + (1 - \alpha_n)d(z_n, Tp) \\ &\le \alpha_n\|x_n - p\| + (1 - \alpha_n)H(Tu_n, Tp) \\ &\le \|x_n - p\|.\end{aligned}$$

Hence $\lim_{n\to\infty}\|x_n - p\|$ exists.

Step 2. Show that $\|z_n - x_n\| \to 0$ as $n \to \infty$ for all $z_n \in Tu_n$. From Lemma 2.1 and T satisfying Condition (A), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - z_n\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)d(z_n, Tp)^2 - \alpha_n(1 - \alpha_n)\|x_n - z_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)H(Tu_n, Tp)^2 - \alpha_n(1 - \alpha_n)\|x_n - z_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - z_n\| \\ &\le \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - z_n\|^2.\end{aligned}$$

This implies that

$$\alpha_n(1-\alpha_n)\|x_n - z_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

From Condition (1) and the existence of $\lim_{n\to\infty}\|x_n - p\|$, we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.5}$$

Step 3. Show that $\|u_n - z_n\| \to 0$ as $n \to \infty$ for all $z_n \in Tu_n$. For any $p \in \Theta$, we estimate

$$\begin{aligned}
\|u_n - p\|^2 &= \bigl\|T_{r_n}^{F_1}\bigl(I - \gamma A^*\bigl(I - T_{r_n}^{F_2}\bigr)A\bigr)x_n - p\bigr\|^2 \\
&= \bigl\|T_{r_n}^{F_1}\bigl(I - \gamma A^*\bigl(I - T_{r_n}^{F_2}\bigr)A\bigr)x_n - T_{r_n}^{F_1}p\bigr\|^2 \\
&\le \bigl\|x_n - \gamma A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n - p\bigr\|^2 \\
&\le \|x_n - p\|^2 + \gamma^2\bigl\|A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\|^2 + 2\gamma\bigl\langle p - x_n, A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|u_n - p\|^2 \le{}& \|x_n - p\|^2 + \gamma^2\bigl\langle Ax_n - T_{r_n}^{F_2}Ax_n, AA^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle \\
&+ 2\gamma\bigl\langle p - x_n, A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle.
\end{aligned} \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}
\gamma^2\bigl\langle Ax_n - T_{r_n}^{F_2}Ax_n, AA^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle &\le L\gamma^2\bigl\langle Ax_n - T_{r_n}^{F_2}Ax_n, Ax_n - T_{r_n}^{F_2}Ax_n\bigr\rangle \\
&= L\gamma^2\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&2\gamma\bigl\langle p - x_n, A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle \\
&\quad = 2\gamma\bigl\langle A(p - x_n), Ax_n - T_{r_n}^{F_2}Ax_n\bigr\rangle \\
&\quad = 2\gamma\bigl\langle A(p - x_n) + \bigl(Ax_n - T_{r_n}^{F_2}Ax_n\bigr) \\
&\qquad - \bigl(Ax_n - T_{r_n}^{F_2}Ax_n\bigr), Ax_n - T_{r_n}^{F_2}Ax_n\bigr\rangle \\
&\quad = 2\gamma\bigl\{\bigl\langle Ap - T_{r_n}^{F_2}Ax_n, Ax_n - T_{r_n}^{F_2}Ax_n\bigr\rangle - \bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2\bigr\} \\
&\quad \le 2\gamma\left\{\frac{1}{2}\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2 - \bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2\right\} \\
&\quad = -\gamma\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2.
\end{aligned} \tag{3.8}$$

Using (3.6), (3.7), and (3.8), we have

$$\begin{aligned}
\|u_n - p\|^2 &\le \|x_n - p\|^2 + L\gamma^2\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2 - \gamma\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2 \\
&= \|x_n - p\|^2 + \gamma(L\gamma - 1)\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2.
\end{aligned} \tag{3.9}$$

It follows that, for all $z_n \in Tu_n$,

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)z_n - p\|^2 \\
&\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|z_n-p\|^2 \\
&= \alpha_n\|x_n-p\|^2 + (1-\alpha_n)d(z_n,Tp)^2 \\
&\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)H(Tu_n,Tp)^2 \\
&\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|u_n-p\|^2 \\
&\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\bigl(\|x_n-p\|^2 + \gamma(L\gamma-1)\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2\bigr) \\
&\le \|x_n-p\|^2 + \gamma(L\gamma-1)\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2.
\end{aligned}$$

Therefore, we have

$$-\gamma(L\gamma-1)\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2.$$

Since $\gamma(L\gamma-1) < 0$ and $\lim_{n\to\infty}\|x_n-p\|$ exists, by (3.8), we obtain

$$\lim_{n\to\infty}\bigl\|Ax_n - T_{r_n}^{F_2}Ax_n\bigr\| = 0. \tag{3.10}$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive and $I-\gamma A^*(T_{r_n}^{F_2}-I)A$ is nonexpansive, it follows that

$$\begin{aligned}
&\|u_n-p\|^2 \\
&\quad= \bigl\|T_{r_n}^{F_1}\bigl(x_n-\gamma A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr) - T_{r_n}^{F_1}p\bigr\|^2 \\
&\quad\le \bigl\langle T_{r_n}^{F_1}\bigl(x_n-\gamma A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr) - T_{r_n}^{F_1}p, x_n-\gamma A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n - p\bigr\rangle \\
&\quad= \bigl\langle u_n-p, x_n-\gamma A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n - p\bigr\rangle \\
&\quad= \frac{1}{2}\bigl\{\|u_n-p\|^2 + \bigl\|x_n-\gamma A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n - p\bigr\|^2 - \bigl\|u_n-x_n-\gamma A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr\|^2\bigr\} \\
&\quad\le \frac{1}{2}\bigl\{\|u_n-p\|^2 + \|x_n-p\|^2 - \bigl\|u_n-x_n-\gamma A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr\|^2\bigr\} \\
&\quad= \frac{1}{2}\bigl\{\|u_n-p\|^2 + \|x_n-p\|^2 - \bigl(\|u_n-x_n\|^2 + \gamma^2\bigl\|A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr\|^2 \\
&\qquad - 2\gamma\bigl\langle u_n-x_n, A^*\bigl(I-T_{r_n}^{F_2}-I\bigr)Ax_n\bigr\rangle\bigr)\bigr\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|u_n-p\|^2 &\le \|x_n-p\|^2 - \|u_n-x_n\|^2 + 2\gamma\bigl\langle u_n-x_n, A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle \\
&\le \|x_n-p\|^2 - \|u_n-x_n\|^2 + 2\gamma\|u_n-x_n\|\bigl\|A^*\bigl(I-T_{r_n}^{F_2}\bigr)Ax_n\bigr\|.
\end{aligned} \tag{3.11}$$

It follows from (3.4) that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|z_n-p\|^2 \quad \text{for all } z_n \in Tu_n
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(z_n, Tp)^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(Tu_n, Tp)^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\bigl(\|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\|u_n - x_n\|\bigl\|A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\|\bigr).
\end{aligned}$$

Therefore, we have

$$(1-\alpha_n)\|u_n - x_n\|^2 \leq 2\gamma\|u_n - x_n\|\bigl\|A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\| + \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

From Condition (1) and (3.10), we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.12}$$

From (3.5) and (3.12), we have

$$\|u_n - z_n\| \leq \|u_n - x_n\| + \|x_n - z_n\| \to 0 \tag{3.13}$$

as $n \to \infty$.

Step 4. Show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. From (3.12) and (3.13), we have

$$\begin{aligned}
\|x_{n+1} - u_n\| &= \bigl\|\alpha_n x_n + (1-\alpha_n)z_n - u_n\bigr\| \\
&\leq \alpha_n\|x_n - u_n\| + (1-\alpha_n)\|z_n - u_n\| \to 0
\end{aligned} \tag{3.14}$$

as $n \to \infty$. From (3.12) and (3.14), we also have

$$\|u_n - u_{n+1}\| \leq \|u_n - x_{n+1}\| + \|x_{n+1} - u_{n+1}\| \to 0 \tag{3.15}$$

as $n \to \infty$. It follows from (3.12) and (3.14) that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_n\| + \|u_n - x_n\| \to 0 \tag{3.16}$$

as $n \to \infty$.

Step 5. Show that $\omega_w(x_n) \subset \Theta$, where $\omega_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x, \{x_{n_i}\} \subset \{x_n\}\}$. Since $\{x_n\}$ is bounded and H is reflexive, $\omega_w(x_n)$ is nonempty. Let $q \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ converging weakly to q. From (3.11), it follows that $u_{n_i} \rightharpoonup q$ as $i \to \infty$. By Lemma 3.2 and (3.13), we obtain $q \in F(T)$.

Next, we show that $q \in EP(F_1)$. From $u_n = T_{r_n}^{F_1}(I + \gamma A^*(I - T_{r_n}^{F_2})A)x_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\bigl\langle y - u_n, u_n - x_n - \gamma A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle \geq 0$$

for all $y \in C$, which implies that

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\bigl\langle y - u_n, \gamma A^*\bigl(I - T_{r_n}^{F_2}\bigr)Ax_n\bigr\rangle \geq 0$$

for all $y \in C$. By Assumption 2.3(2), we have

$$\frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle - \frac{1}{r_{n_i}}\bigl\langle y - u_{n_i}, \gamma A^*\bigl(I - T_{r_{n_i}}^{F_1}\bigr)Ax_{n_i}\bigr\rangle \geq F_1(y, u_{n_i})$$

for all $y \in C$. From $\liminf_{n\to\infty} r_n > 0$, from (3.10), (3.12), and Assumption 2.3(4), we obtain

$$F_1(y,q) \leq 0$$

for all $y \in C$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$, $y_t \in C$, and hence $F_1(y_t, q) \leq 0$. So, by Assumption 2.3(1) and (4), we have

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, q) \leq tF_1(y_t, y)$$

and hence $F_1(y_t, y) \geq 0$. So $F_1(q, y) \geq 0$ for all $y \in C$ by (3.15) and hence $q \in EP(F_1)$. Since A is a bounded linear operator, $Ax_{n_i} \rightharpoonup Aq$. Then it follows from (3.10) that

$$T_{r_{n_i}}^{F_2}Ax_{n_i} \rightharpoonup Aq \tag{3.17}$$

as $i \to \infty$. By the definition of $T_{r_{n_i}}^{F_2}Ax_{n_i}$, we have

$$F_2\bigl(T_{r_{n_i}}^{F_2}Ax_{n_i}, y\bigr) + \frac{1}{r_{n_i}}\bigl\langle y - T_{r_{n_i}}^{F_2}Ax_{n_i}, T_{r_{n_i}}^{F_2}Ax_{n_i} - Ax_{n_i}\bigr\rangle \geq 0$$

for all $y \in C$. Since F_2 is upper semi-continuous in the first argument and (3.17), it follows that

$$F_2(Aq, y) \geq 0$$

for all $y \in C$. This shows that $Aq \in EP(F_2)$. Hence $q \in \Omega$.

Step 6. Show that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of Θ. It is sufficient to show that $\omega_w(x_n)$ is single point set. Let $p, q \in \omega_w(x_n)$ and $\{x_{n_k}\}, \{x_{n_m}\} \subset \{x_n\}$ be such that $x_{n_k} \rightharpoonup p$ and $x_{n_m} \rightharpoonup q$. From (3.12), we also have $u_{n_k} \rightharpoonup p$ and $u_{n_m} \rightharpoonup q$. By Lemma 3.2 and (3.13), it follows that $p, q \in F(T)$. Applying Lemma 2.2, we obtain $p = q$. This completes the proof. □

If $Tp = \{p\}$ for all $p \in F(T)$, then T satisfies Condition (A) and so we can obtain the following result.

Theorem 3.4 *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to K(C)$ a $\frac{1}{2}$-nonspreading multi-valued mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption* 2.3 *and F_2 is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$, where $\Omega = \{z \in C : z \in EP(F_1)$ and $Az \in EP(F_2)\}$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} \in \alpha_n x_n + (1-\alpha_n)Tu_n, \end{cases} \tag{3.18}$$

for all $n \geq 1$, *where* $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0,\infty)$, *and* $\gamma \in (0,1/L)$ *such that* L *is the spectral radius of* A^*A *and* A^* *is the adjoint of* A. *Assume that the following conditions hold*:

(1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(2) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ *defined by* (3.18) *converges weakly to* $p \in \Theta$.

Since P_T satisfies Condition (A), we also obtain the following results.

Theorem 3.5 *Let* H_1, H_2 *be two real Hilbert spaces and* $C \subset H_1$, $Q \subset H_2$ *be nonempty closed convex subsets of Hilbert spaces* H_1 *and* H_2, *respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator and* $T : C \to P(C)$ *a multi-valued mapping. Let* $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ *be bifunctions satisfying Assumption* 2.3 *and* F_2 *is upper semi-continuous in the first argument. Assume that* P_T *is* $\frac{1}{2}$*-nonspreading multi-valued mapping and* $I - T$ *is demiclosed at* 0 *with* $\Theta = F(T) \cap \Omega \neq \emptyset$, *where* $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. *Let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} \in \alpha_n x_n + (1-\alpha_n)P_T u_n, \end{cases} \tag{3.19}$$

for all $n \geq 1$, *where* $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0,\infty)$, *and* $\gamma \in (0,1/L)$ *such that* L *is the spectral radius of* A^*A *and* A^* *is the adjoint of* A. *Assume that the following conditions hold*:

(1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(2) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ *defined by* (3.19) *converges weakly to* $p \in \Theta$.

Proof By the same proof as in Theorem 3.3, we have $u_n \to z_n \in P_T u_n$. This implies that

$$d(u_n, Tu_n) \leq d(u_n, P_T u_n) \leq \|u_n - z_n\| \to 0 \tag{3.20}$$

as $n \to \infty$. Since $I - T$ is demiclosed at 0, we obtain this result. □

4 Examples and numerical results

In this section, we give examples and numerical results for supporting our main theorem.

Example 4.1 Let $H_1 = H_2 = \mathbb{R}$, $C = [-3,0]$, and $Q = (-\infty,0]$. Let $F_1(u,v) = (u-1)(v-u)$ for all $u,v \in C$ and $F_2(x,y) = (x+15)(y-x)$ for all $x,y \in Q$. Define two mappings $A : \mathbb{R} \to \mathbb{R}$ and $T : C \to K(C)$ by $Ax = 3x$ for all $x \in \mathbb{R}$ and

$$Tx = \begin{cases} \{0\}, & x \in [-2,0]; \\ [-\frac{|x|}{|x|+1}, 0], & x \in [-3,-2). \end{cases}$$

Choose $\alpha_n = \frac{n}{2n+1}$, $r_n = \frac{n}{n+1}$, and $\gamma = \frac{1}{10}$. It is easy to check that F_1 and F_2 satisfy all conditions in Theorem 3.3 and T satisfies Condition (A) such that $F(T) = \{0\}$. For each $r > 0$ and $x \in C$, we divide the process of our iteration into five steps as follows:

Table 1 Numerical results of Example 1.1 being randomized in the first time

n	z_n	x_n	$\lvert x_{n+1} - x_n\rvert$
1	−5.74683E−01	−3.00000E+00	1.61688E+00
2	0	−1.38312E+00	8.29873E−01
3	0	−5.53249E−01	3.16142E−01
4	0	−2.37107E−01	1.31726E−01
5	0	−1.05381E−01	5.74804E−02
6	0	−4.79003E−02	2.57925E−02
7	0	−2.21078E−02	1.17909E−02
8	0	−1.03170E−02	5.46194E−03
9	0	−4.85506E−03	2.55529E−03
10	0	−2.29976E−03	1.20464E−03
⋮	⋮	⋮	⋮
50	0	−9.26099E−16	4.67634E−16

Step 1. Find $z \in Q$ such that $F_2(z,y) + \frac{1}{r}\langle y-z, z-Ax\rangle \geq 0$ for all $y \in Q$. Noting that $Ax = 3x$, we have

$$\begin{aligned} F_2(z,y) + \frac{1}{r}\langle y-z, z-Ax\rangle \geq 0 \quad &\Longleftrightarrow \quad (z+15)(y-z) + \frac{1}{r}\langle y-z, z-3x\rangle \geq 0 \\ &\Longleftrightarrow \quad r(z+15)(y-z) + (y-z)(z-3x) \geq 0 \\ &\Longleftrightarrow \quad (y-z)\big((1+r)z - (3x-15r)\big) \geq 0. \end{aligned}$$

By Lemma 2.4, we know that $T_r^{F_2}Ax$ is single-valued. Hence $z = \frac{3x-15r}{1+r}$.

Step 2. Find $s \in C$ such that $s = x - \gamma A^*(I - T_r^{F_2})Ax$. From Step 1, we have

$$\begin{aligned} s &= x - \gamma A^*\big(I - T_r^{F_2}\big)Ax = x - \gamma A^*\big(Ax - T_r^{F_2}Ax\big) \\ &= x - \gamma\left(9x - \frac{3(3x-15r)}{1+r}\right) \\ &= (1-9\gamma)x + \frac{3\gamma}{1+r}(3x-15r). \end{aligned}$$

Step 3. Find $u \in C$ such that $F_1(u,v) + \frac{1}{r}\langle v-u, u-s\rangle \geq 0$ for all $v \in C$. From Step 2, we have

$$\begin{aligned} F_1(u,v) + \frac{1}{r}\langle v-u, u-s\rangle \geq 0 \quad &\Longleftrightarrow \quad (u-1)(v-u) + \frac{1}{r}\langle v-u, u-s\rangle \geq 0 \\ &\Longleftrightarrow \quad r(u-1)(v-u) + (v-u)(u-s) \geq 0 \\ &\Longleftrightarrow \quad (v-u)\big((1+r)u - (s+r)\big) \geq 0. \end{aligned}$$

Similarly, by Lemma 2.4, we obtain $u = \frac{s+r}{1+r} = \frac{(1-9\gamma)x+r}{1+r} + \frac{3\gamma(3x-15r)}{(1+r)^2}$.

Step 4. Find $x_{n+1} \in \alpha_n x_n + (1-\alpha_n)Tu_n$, where $u_n = \frac{(1-9\gamma)x_n+r_n}{1+r_n} + \frac{3\gamma(3x_n-15r_n)}{(1+r_n)^2}$. From

$$Tx = \begin{cases} \{0\}, & x \in [-2,0]; \\ [-\frac{|x|}{|x|+1}, 0], & x \in [-3,-2), \end{cases}$$

and $\alpha_n = \frac{n}{2n+1}$, $r_n = \frac{n}{n+1}$, and $\gamma = \frac{1}{10}$, we have

$$x_{n+1} = \left(\frac{n}{2n+1}\right)x_n + \left(1 - \frac{n}{2n+1}\right)z_n, \tag{4.1}$$

Table 2 Numerical results of Example 1.1 being randomized in the second time

n	z_n	x_n	$\vert x_{n+1} - x_n\vert$
1	−1.26779E−00	−3.00000E+00	1.91548E+00
2	0	−1.08452E+00	6.50711E−01
3	0	−4.33808E−01	2.47890E−01
4	0	−1.85918E−01	1.03288E−01
5	0	−8.26300E−02	4.50709E−02
6	0	−3.75591E−02	2.02241E−02
7	0	−1.73350E−02	9.24532E−03
8	0	−8.08965E−03	4.28276E−03
9	0	−3.80690E−03	2.00363E−03
10	0	−1.80327E−03	9.44568E−04
⋮	⋮	⋮	⋮
50	0	−7.26163E−16	3.66677E−16

where

$$z_n \in \begin{cases} \{0\}, & u_n \in [-2,0]; \\ [-\frac{|u_n|}{|u_n|+1}, 0], & u_n \in [-3,-2). \end{cases}$$

Step 5. Compute the numerical results. Choosing $x_1 = -3$ and taking randomly z_n in the above interval, we obtain Tables 1 and 2.

From Table 1 and Table 2, we see that 0 is the solution in Example 1.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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